CEMC GRADES 9/10 MATH CIRCLES
NOVEMBER 23/30, 2022
FORMAL LOGIC

1. Introduction

Formal Logic has existed since antiquity. As long have there been people around to ask the question "Why?", they have sought to define how we can use reasoning to make deductions. Every person on earth uses logic in their day to day lives when making decisions. Computer Scientists use logic when they program computers to ensure their code does what they want it to. Politicians use logic when they are debating another politician and analyzing their statements for lies. It could be as simple as deciding which route to take to school/work. But not everyone makes the correct decisions all the time. Using formal logic we can analyze where others made mistakes in their reasoning, and ensure that we never make mistakes ourselves. This reasoning will be very helpful when trying to solve math problems.

1.1. **Exercise.** Try and find where the mistake is in this argument. (You may assume each item in the list is factual)

(1) Someone who plays fortnite is cool
(2) I do not play fortnite.

Therefore I am uncool.

The issue is that we only know something about people who *do* play fortnite. There might be other reasons why someone could be cool. If we knew that *only* people who play fortnite are cool, then anyone who doesn't play it is automatically uncool.

For those who do the CEMC math contests, knowing how to correctly reason is essential for solving problems like the one below.

1.2. **Example.** (Taken from 2022 Pascal Contest) Dhruv is older than Bev. Bev is older that Elcim. Elcim is younger than Andy. Andy is younger than Bev. Bev is younger than Cao. Who is the third oldest?

What we would like to do is figure out what it means for an argument to be valid, and come up with a systematic way to check the validity of arguments. However, it is very difficult to find a system that will work for *every* argument as there are many ways to say the same thing. Therefore we will build up a set of rules from very easy statements and then find out how to transform any statement into one of that form.

After you finish reading each section feel free to try the exercises from the same section of the problem set.

2. Implications

An implication is at the heart of reasoning. The best way to learn what it means for one statement to imply another is to see an example.

2.1. **Example.** If it is raining then the ground is wet outside.

All implications can be stated in the form "If P then Q" where P and Q are just statements themselves. In this framework P is called the hypothesis, and Q is the conclusion. To save space we will write $P \Rightarrow Q$ to mean "If P then Q".

2.2. **Stop and Think.** You may have heard the word "hypothesis" mentioned with respect to scientists. Can the hypotheses that they are testing also be stated in the form "If P then Q"?
Take the famous example "Smoking cigarettes increases your risk of lung cancer" and transform it into an implication.

In Example 2.1 we are not saying anything about the current weather outside, we are just saying that *if* it is raining then the ground outside will be wet (don't worry about what "the ground" means).

What if it is not raining? Do we know anything about the truth of the statement "the ground outside is wet"? The answer is NO. If it is sunny outside there could still be many reasons why the ground outside is wet.

When we use $P, Q, R, \ldots$ to represent statements from now on we will assume that they are *atomic* statements. These are very simple statements that must be either true or false such as "It is raining". We call the symbolic representation $P, Q, R, \ldots$ *atomic formulas*.

2.3. **Stop and Think.** We use the word "atomic" because, just like atoms, atomic statements are the building blocks of all larger statements.

Any formal statement can be broken down into its atoms and its truth is completely determined by the truth values of its atoms. Since we now know what an implication is, let us consider its truth in relation to the truth of its atoms. When we are dealing with a statement composed of many atoms, there can be many possibilities, and so we summarize them in a *truth table* where each row represents a possibility and each column is a statement.

2.4. **Example.** (Note that T =True, F =False)

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

What the above example tells us is that the only time an implication is false is when the hypothesis is true and the conclusion is false. This should line up with what we expect to be the case. In our earlier example we have P being the statement "It is raining" and Q is "The ground is wet outside". If it is raining and the ground is NOT wet outside (like the scenario in the second row of our table), then our original statement "If it is raining then the ground is wet outside" must be wrong. The third row of the table corresponds to the possibility of it not raining outside and the ground still being wet, which we said was alright. This truth table holds true for *any* implication, whether or not the statement itself is sensible.

Statements of the form "If P then Q" are used extensively in computer science. If you want to write password protection for a website you are building, then you will need to include code that says something like "If the user enters the correct password then let them in to their account".

3. Other Logical Connectives

The implication symbol $\Rightarrow$ is called a *logical connective* because it is used to connect two atomic formulas into a more complicated one. Not every statement that we would like to look at can be expressed using "If... then..." however, so we will now introduce more logical connectives and consider their truth tables.

The symbol $\wedge$ is used to connect two atomic statements where we would use the English word "and".

3.1. **Example.** If $P =$ "I am going buy eggs" and $Q =$"I am going to buy milk". Then the formula

$$P \wedge Q$$

says that "I am going to buy eggs AND milk".

Let us figure out what the truth table should be for *any* formula of the form $P \wedge Q$ by looking at our example. If I go to the store and I do not buy anything, then clearly the statement "I am going to buy eggs AND milk" is false. If I instead only buy milk and not eggs, then I still have not fulfilled my promise to buy eggs and milk. The only way that this example can be true is if I buy *both* eggs and milk. Therefore the only row of our truth table for which the statement $P \wedge Q$ should be true is if *both* P and Q are true. The truth table is shown below

3.2. **Example.**

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Now that we know two connectives we can start using them together to make much more complicated formulas. We use brackets so that there is no confusion in our formulas. The formula $(P \wedge Q) \Rightarrow R$ represents the statement

"If P is true and Q is true then R is true", but the formula $P \wedge (Q \Rightarrow R)$ represents "P is true and if Q is true then R is true". Without the brackets we wouldn't know which way to read $P \wedge Q \Rightarrow R$.

3.3. **Exercise.** Can you fill in the missing entries in the following truth table

P	Q	$P \wedge Q$	$Q \Rightarrow (P \wedge Q)$
T	T	T	T
T	F		T
F	T	F	F
F	F	F	

The next symbol that we are going to learn is $\vee$ which is used to mean "or". The logical $\vee$ differs slightly from the way that you might use "or" in your regular life because the logical $\vee$ includes the possibility of both. It is called an "inclusive or", and it will be explained by an example below.

3.4. **Example.** Let $P =$"I will go to work today" and $Q =$"I will go to work tomorrow". Then the formula

$$P \vee Q$$

says that "I will go to work today or tomorrow".

In this example If I go to work both today *and* tomorrow then I have not broken my promise. Using an inclusive "or" means that P is true, or Q is true, or both. The only way that an or statement is false is if both parts are false. Thus the truth table looks like our example below.

3.5. **Example.**

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

The last logical symbol that we will learn is the negation symbol $\neg$ which roughly translates to the word "not". It is different from all of the others because it is not a connective. It is applied to a single formula and it reverses all truth values that formula might have. If $P =$"I can ride a bicycle", then $\neg P =$"I cannot ride a bicycle". It is rather easy to see that if it is true that I can ride a bike, then it is false to say that I cannot ride a bike. The truth table for a formula using this symbol is therefore very simple and shown below.

3.6. **Example.**

P	$\neg P$
T	F
F	T

With just these four symbols and our atomic formulas we can now express a wide variety of statements as logical formulas, and knowing how the truth tables for each symbol work we can determine the truth values of these formulas.

3.7. **Example.** The formula $P \wedge \neg Q$ is only true if P is true and $\neg Q$ is true. $\neg Q$ is only true when Q is false. Thus the truth table looks like

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

3.8. **Exercise.** Can you write a formula that represents the *exclusive* or? Given P, Q we want a formula that is true when P is true or Q is true, but not both. Think about how you would say such a formula using the words "and, or, not".

Observe the following two truth tables

3.9. **Example.**

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

3.10. **Stop and Think.** What do you notice about these two tables?

The last column of both tables is the same! If two formulas are true at exactly the same times then we consider them to be the same formula and call them *truth equivalent.* Two truth equivalent formulas may be treated as the same formula when the only thing that we need from a formula is it's truth values. This just shows that there can be multiple correct ways to say the same thing.

4. Tautologies and how to win arguments

We now have all of the tools we need to analyze a wide variety of arguments and check whether they are logically sound. We just need to know how to check. For this we need a very special type of formula. A formula is called a *tautology* if it is true in any circumstance. If you have been doing the exercises up to this point you will have already seen an example of a tautology, but they can have many forms.

4.1. **Stop and Think.** Try and come up with your own tautology. Make a statement that is always true, regardless of the truth of its atomic components. (Hint: It cannot be atomic)

Consider the statement "I'll clean it if I clean it". Putting it in our recently established logical framework we can see that it's the same as saying "If I clean it then I will clean it". Letting $P =$"I will clean it", the formula is $P \Rightarrow P$.

4.2. **Execise.** Construct the truth table for $P \Rightarrow P$.

Anytime we give an argument we are stating some assumptions and arriving at a conclusion. For that conclusion to be valid given the assumptions we want the implication "If (assumptions) then (conclusion)" to always be true. It is also true that if an implication is always true, then the argument is valid. So to check our arguments we want to transform them into logical formulas. Make all of the assumptions our hypothesis, and see if the implication is a tautology. Below we shall see an example.

4.3. **Example.** Consider the argument

(1) Every time I walk past this dog it barks at me.
(2) I am walking past this dog.

Therefore it will bark at me.

Hopefully by now we can see that this example argument is sound, but let us prove this. There are two atomic components of the statement, so we let $P =$"I walk past this dog" and $Q =$"It will bark at me". The first assumption that we made was that if I walk past the dog then it will bark at me, which is $P \Rightarrow Q$. The second assumption is P. We are assuming both though, so our hypothesis is $(P \Rightarrow Q) \wedge P$. The more assumptions that you make the more we will have to combine using $\wedge$. The conclusion that we hope is reached is Q, so we test the argument $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ and consider its truth table.

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge P$	$((P \Rightarrow Q) \wedge P) \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Thus the argument in Example 4.3 is a valid one. It is an example of a more general rule called Modus Ponens, which says that $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ is a tautology for any P and Q.

5. Semantics

So far we have seen how to use propositional formulas to check whether or not an argument that we have made is correct. This type of logic is most useful to those interested in math to check the validity of proofs that we have made. However to check if a proof is correct we first have to think of a proof. Logic can help us in this endeavour as well.

We have been focusing mostly on how to write the formulas so that they say what we want them to. This is called the *syntax* of our language. However we can say the same thing in many different ways. Think back to Example 3.9 where we saw that the two formulas $\neg(P \vee Q)$ and $\neg P \vee \neg Q$ have the exact same truth tables. This means that as far as their meaning is concerned, they are the same, and so we call them logically equivalent and write $\neg(P \vee Q) \equiv \neg P \vee \neg Q$. We are now not just looking at arrangements of symbols in the formulas (its syntax), but at the meaning of the formulas (its *semantics*).

5.1. **Example.** Let $P =$"An integer is a multiple of 2", $Q =$"An integer is a multiple of 3", and $R =$"An integer is a multiple of 6".
Prove that $(\neg P \vee \neg Q) \Rightarrow \neg R$. (There are many ways to do this.)

We are going to use the logical equivalence above and instead show that $\neg(P \wedge Q) \Rightarrow \neg R$. Notice that $(P \wedge Q)$ says that an integer is a multiple of 2 and 3. Anything that is a multiple of 2 and 3 is automatically a multiple of 6, so $\neg(P \wedge Q)$ is the same as saying that an integer is not a multiple of 6. From this it is easier to see why $\neg(P \wedge Q) \Rightarrow \neg R$.

This example is a little forced, but hopefully shows that we can use logical equivalences to simplify the statements of problems, so that a proof is easier to find. If we can replace a part of a formula by something that is syntactically simpler and doesn't change the meaning then we generally should do so. But finding a logically equivalent formula is no easy feat, so we are going to establish some common examples that we will call *logical identities*, and use these identities instead. The first one we have already seen in Example 3.9. A very similar example is given below.

5.2. **Example.**

$$\neg P \wedge \neg Q \equiv \neg(P \vee Q)$$

5.3. **Exercise.** Check that these formulas are actually logically equivalent by writing their truth tables.

These first two identities should hopefully be intuitive to the reader. As an example of the one above, if I were to say "It is not Tuesday or Wednesday" then certainly it is not Tuesday *and* it is not Wednesday. These identities can reduce the number of negations in formulas, but they do not change the number of occurrences of propositional variables, so the formulas that we are writing do not get much simpler. That being said, they are incredibly important identities and should be committed to memory. Let us see examples of identities that will shrink our formulas in a bigger way.

5.4. **Example.**

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

5.5. **Example.**

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

5.6. **Stop and Think.** Try coming up with an example of one of the above identities to see if it makes sense. If it does not then you can always write the truth table.

Let us see an example of how these new identities can be used to simplify problems that we may encounter in math.

5.7. **Example.** De Morgans Law

The next identity that we will see is so important that it gets it's own section.

6. Contrapositive

6.1. **Example.**

$$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$$

Almost every problem that one will encounter in math will use an implication somewhere in it. We will have some hypotheses and try to show that if we assume they are true, then the conclusion follows. What the above identity tells us is that we can instead assume the negation of the conclusion and show that the negation of the hypothesis is implied. We will see an example of when this can make a proof simpler below.

6.2. **Example.** Prove that for an integer x, if x^2 is even then x is even.

The contrapositive of this statement is "If x is not even then x^2 is not even. We know that an integer that is not even is odd, and two odd numbers multiplied together is odd, so the proof follows immediately.

Proof by contrapositive is a strong technique because we can select the part of the implication that gives us the most hypotheses to use in our proof. Notice that $P \Rightarrow \neg(Q \vee R \vee S) \equiv (\neg Q \wedge \neg R \wedge \neg S) \Rightarrow \neg P$ by applying two of our new identities. To prove The original statement $P \Rightarrow \neg(Q \vee R \vee S)$ we need to assume P and show that none of Q, R, S hold. This would involve checking many cases. If we were to do a proof by contrapositive we would assume that Q, R, and S are all false, and then show that P must be false. This gives us more assumptions to work with and fewer things to have to prove.

We will wrap up this section by looking at other forms of implications and showing which are logically equivalent.

6.3. **Example.**

P	Q	$P \Rightarrow Q$	$\neg P \Rightarrow \neg Q$	$Q \Rightarrow P$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

The formula $\neg P \Rightarrow \neg Q$ from the fourth column of the table is called the *inverse* of $P \Rightarrow Q$ and we can see that it is not logically equivalent to $P \Rightarrow Q$, but it is logically equivalent to $Q \Rightarrow P$, which we call the *converse*. This should not be surprising because $Q \Rightarrow P$ is not the same as saying $P \Rightarrow Q$. "If it is raining outside then the ground is wet", is different from "If the ground outside is wet then it is raining".

6.4. **Exercise.** What is the contrapositive of the inverse of $P \Rightarrow Q$?

From the table we can also see that the truth values of $\neg P \Rightarrow \neg Q$ are not the exact opposites of $P \Rightarrow Q$ which might make one think "What is $\neg(P \Rightarrow Q)$?". This is a very special identity and will get its own section.

7. Reductio ad absurdum

7.1. **Example.**

P	Q	$P \Rightarrow Q$	$\neg P \vee Q$	$P \wedge \neg Q$
T	T	T	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

The important identity that we get from the above table is that $P \Rightarrow Q \equiv \neg P \vee Q$. In our common example $\neg P \vee Q$ says "It isn't raining or the ground outside is wet". This OR statement is only false when it is raining and the ground outside is *not* wet. This lines up with our original implication. $P \Rightarrow Q$ is the same as saying P is not true (in which case we do not care about Q) or if P is true then Q must be true.

Notice that with this latest identity we could get rid of all implications in a formula and turn it into something that only uses $\vee$, $\wedge$, and $\neg$. This would make it very easy to check what truth values satisfy the formula, because we just make sure that everything with a negation is false and everything without one is true. This is at the heart of the Boolean Circuit Satisfiability Problem which anyone who is interested is encouraged to read about.

We are going to use this identity for another purpose however. We would like to prove an implication $P \Rightarrow Q$ is true, and this can be done by instead showing that $\neg(P \Rightarrow Q)$ is false. What we also get from the above table is that $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$. A proof method called "Proof by Contradiction" or "Reductio ad absurdum" (Latin for "Reduction to Absurdity") is where we do exactly that. We take the negation of the implication by assuming P AND $\neg Q$ and try to reach something that is clearly absurd or a contradiction. If the negation of the formula is false, then the original formula must be true! Let's see an example of how this can be useful.

7.2. **Example.** If we draw a square entirely inside of a circle, then the side length of the square must be less than the diameter of the circle.

Let us assume that we have drawn a square inside a circle AND that the side length of the square is greater than or equal to the diameter of the circle. If we look at opposite corners of the square then they are farther apart than the side length which means that they are more than the diameter of the circle apart. But we have also assumed that the whole square is inside of the circle, so certainly every corner is in the circle.

These two fact contradict one another, because the diameter of a circle is the farthest distance between any two points in the circle, but we have assumed that the two opposite corners of the square are both in the circle AND that they are more than the diameter apart. This must mean that our assumption is false and it cannot be the case that a square is inside of a circle and the side length is greater than or equal to the diameter. Thus, if a square is drawn inside of a circle, the side length must be less than the diameter.

The human brain is very good at noticing when something makes sense. Therefore the method of proving a statement by contradiction is a helpful one when stuck trying to prove a statement directly. We will return to proving statements by contradiction once we learn some new types of formulas on which to use it, but for now we end the section with some helpful identities that can clean up redundancies in our formulas.

7.3. **Example.**

$$\neg\neg P \equiv P \quad \text{and} \quad (P \vee P) \equiv (P \wedge P) \equiv P$$

7.4. **Exercise.** Use any of the identities that we have learned to write the following formula using as few symbols as possible.
(Hint: you should only need 3)

$$((\neg P \Rightarrow Q) \wedge \neg(\neg P \wedge \neg Q))$$

8. Conclusion

If you have read the document this far and done all of the exercises then you know more than enough logic to apply it in almost any career setting. But if you are still curious about *why* mathematics works the way it does, and the limits of what can be known/proven then perhaps studying logic in university is for you.